Cosmological constraints on Brans-Dicke theory:

a covariant and gauge-invariant approach

Feng-Quan Wu and Xue-Lei Chen

National Astronomical Observatories, Beijing

Dark Matter and Dark Energy Group

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(Ⅰ)
Brans-Dicke Theory

Brans-Dicke Theory

Action of Brans-Dicke theory in the Jordan frame:

$$\mathcal{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[-\varphi R + \frac{\omega}{\varphi}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi\right] + \mathcal{S}^{(m)}$$

$$G_{eff}(\varphi) = \frac{G}{\varphi}$$

When $\omega \to \infty$, General Relativity is recovered.

(Ⅱ)
Covariant and gauge-invariant approach

Geometrical method

Historical Development

Metric approach:

(1) Lifshitz(1945,1963)-----synchronous gauge:

Gauge problem

(2) Bardeen(1980)-----gauge-invariant variables

Historical Development

Geometrical approach:

Hawking(1966)

....

Ellis & Bruni(1989)

....

Using

(1) 1+3 split of the Bianchi and Ricci identities
(2) kinematic quantities
(3) energy-momentum tensors of the fluid(s)
(4) gravito-electromagnetic parts of the Weyl tensor

Historical Development

Geometrical approach:

Advantage:

(1) clear physical meaning of every quantity

(2) unified treatment of scalar, vector and tensor modes

(3) exact equations

Spacetime Splitting

Introduce timelike 4-velocity field

$$u_a$$

Spatial projection tensor:

$$h_{ab} \equiv g_{ab} - u_a u_b$$

Timelike and Spacelike Derivative

Covariant time derivative:

$$\dot{T}^{b\ldots c}_{d\ldots e} \equiv u^a \nabla_a T^{b\ldots c}_{d\ldots e}$$

Spatial covariant derivative:

$$D^a T^{b\ldots c}{}_{d\ldots e} \equiv h^a_p h^b_r \ldots h^c_s h^t_d \ldots h^u_e \nabla^p T^{r\ldots s}{}_{t\ldots u}$$

Density Perturbation Variable

Density perturbation variable in metric method:

$$\delta = \frac{\delta \rho}{\rho}$$

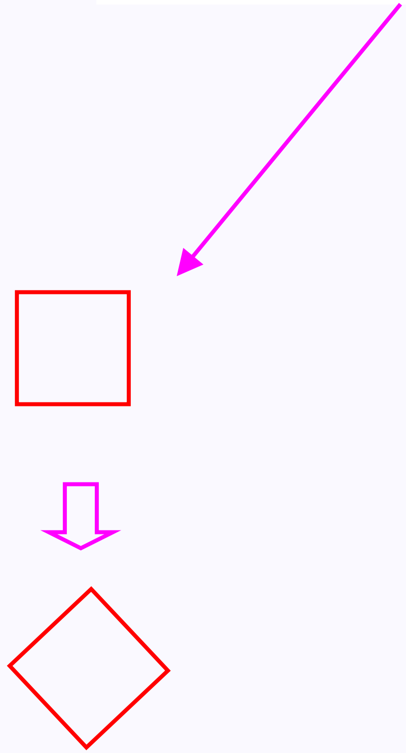
Fractional projected gradient of the density field:

$$\delta = \frac{\delta \rho}{\rho} \quad \Rightarrow \quad X_a \equiv \frac{D_a \rho}{\rho}$$

where $D_a = h_a^b \nabla_b$

Kinematics

$$\nabla_a u_b = \varpi_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} + u_a A_b$$



$$\varpi_{ab} = \mathrm{D}_{[b}u_{a]}$$

Kinematics

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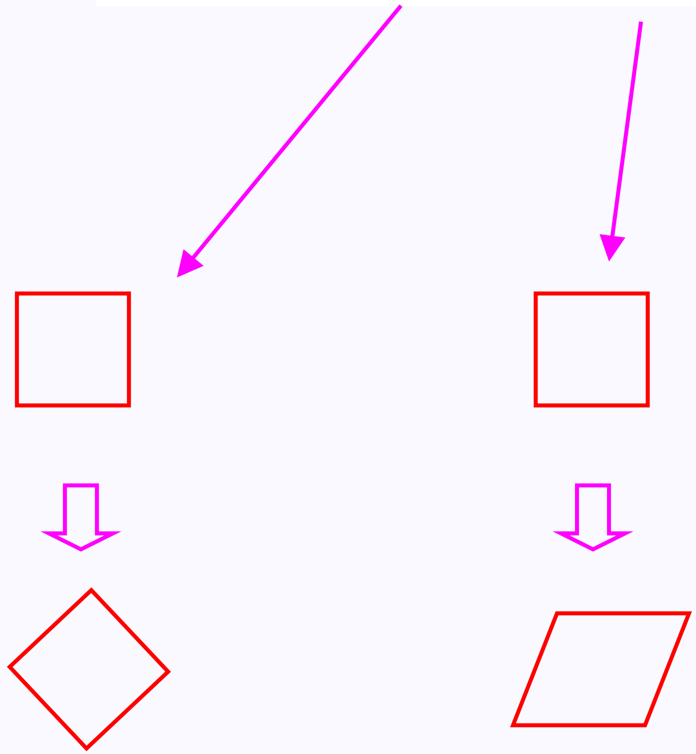
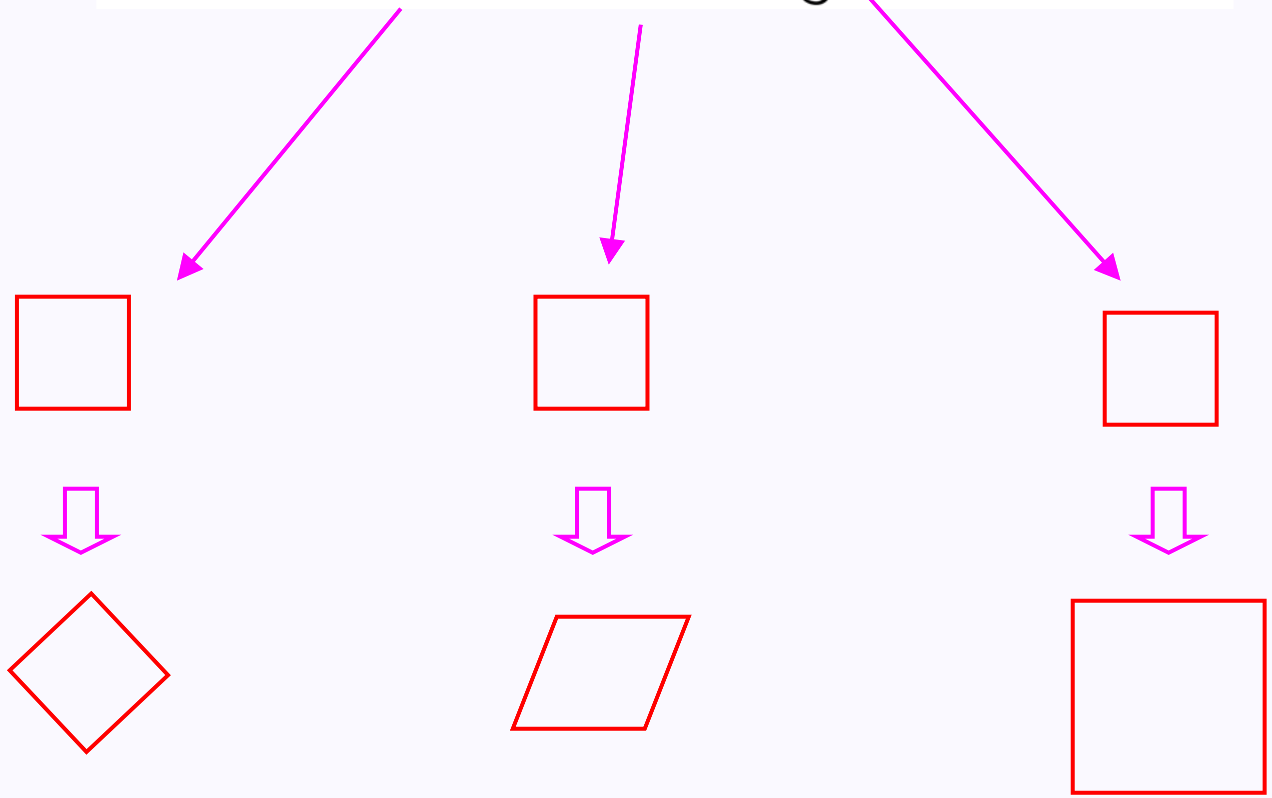


$$\varpi_{ab} = \mathrm{D}_{[b}u_{a]} \qquad \sigma_{ab} = \mathrm{D}_{\langle a}u_{b\rangle}$$

Kinematics

$$\nabla_a u_b = \varpi_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} + u_a A_b$$

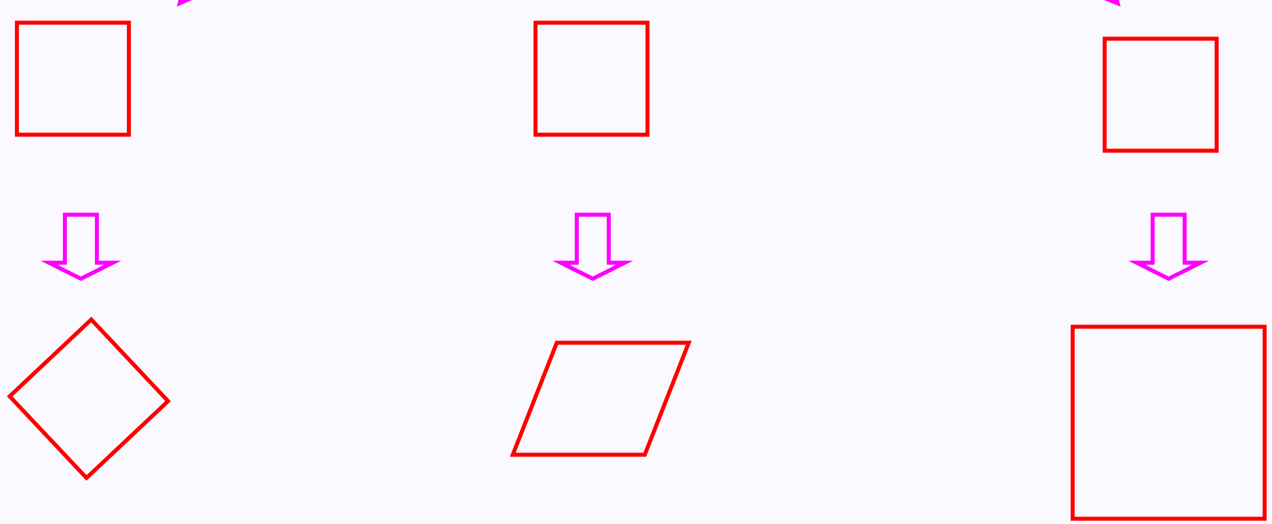


$\varpi_{ab} = \mathsf{D}_{[b}u_{a]}$ $\sigma_{ab} = \mathsf{D}_{\langle a}u_{b\rangle}$ $\theta \equiv \nabla^a u_a = D^a u_a = 3H$

Kinematics

$$\nabla_a u_b = \varpi_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} + u_a A_b$$

$$A_a \equiv u^b \nabla_b u_a = \dot{u}_a$$



$$\varpi_{ab} = \mathsf{D}_{[b} u_{a]}$$

$$\sigma_{ab} = \mathsf{D}_{\langle a} u_{b\rangle}$$

$$\theta \equiv \nabla^a u_a = D^a u_a = 3H$$

Spacetime Curvature

The spacetime decomposition of the Riemann tensor:

$$R_{abcd} = C_{abcd} + \frac{1}{2}\left(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}\right) - \frac{1}{6}R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$$

Spacetime Curvature

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$$C_{abcd} = \left(g_{abqp}g_{cdsr} - \eta_{abqp}\eta_{cdsr}\right)u^q u^s E^{pr} - \left(\eta_{abqp}g_{cdsr} + g_{abqp}\eta_{cdsr}\right)u^q u^s B^{pr}$$

where $g_{abcd} = g_{ac}g_{bd} - g_{ad}g_{bc}$

Kinematical Evolution

Ricci identities

$$2\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d$$

Kinematical Evolution

Ricci identities

$$2\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d$$

$$\Uparrow$$

$$\nabla_a u_b = \varpi_{ab} + \sigma_{ab} + \tfrac{1}{3}\theta h_{ab} + u_a A_b$$

Kinematical Evolution

Ricci identities

$$2\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d$$

$$R_{abcd} = C_{abcd} + \frac{1}{2}\left(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}\right) - \frac{1}{6}R\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$$

$$C_{abcd} = \left(g_{abqp}g_{cdsr} - \eta_{abqp}\eta_{cdsr}\right)u^q u^s E^{pr} + \left(\eta_{abqp}g_{cdsr} + g_{abqp}\eta_{cdsr}\right)u^q u^s B^{pr}$$

Kinematical Evolution

Raychaudhuri equation:

$$\dot{\theta} + \frac{1}{3}\theta^2 - D^a \dot{u}_a + \frac{\kappa}{2\varphi}(\rho + 3P) + \frac{1}{2}\left(2\omega\frac{\dot{\varphi}^2}{\varphi^2} - \frac{GV(\varphi)}{\varphi} + \frac{1}{\varphi}D_a D^a \varphi + \theta\frac{\dot{\varphi}}{\varphi} + 3\frac{\ddot{\varphi}}{\varphi}\right) = 0$$

Vorticity propagation equation:

$$\dot{\omega}_{ab} - D_{[a}\dot{u}_{b]} + \frac{2}{3}\theta\omega_{ab} = 0$$

Shear propagation equation:

$$\dot{\sigma}_{<ab>} + \frac{2}{3}\theta\sigma_{ab} - D_{<a}\dot{u}_{b>} + E_{ab} + \frac{\kappa}{2}\frac{\pi_{ab}}{\varphi} + \frac{1}{2\varphi}D_{<b}D_{a>}\varphi + \frac{1}{2}\frac{\dot{\varphi}}{\varphi}\sigma_{ab} = 0$$

Kinematical Evolution

$$2\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d$$

Shear constraint:

$$D^b\omega_{ab} + D^b\sigma_{ab} - \frac{2}{3}D_a\theta - \frac{\kappa}{\varphi}q_a - \omega\frac{\dot{\varphi}}{\varphi^2}D_a\varphi - \frac{1}{\varphi}(D_a\varphi)^{\cdot} - \frac{\dot{\varphi}}{\varphi}\dot{u}_a = 0$$

Vorticity divergence identity:

$$D^c(\varepsilon_{abc}\omega^{ab}) = 0$$

B_{ab} equation:

$$B_{ab} + (D^c\omega_{d(a} + D^c\sigma_{d(a})\eta_{b)ce}{}^{d}u^e = 0$$

Conservation Laws

Bianchi identities of the Riemann tensor:

$$\nabla_{[e}R_{cd]ab} = 0$$

$$\Downarrow$$

$$\nabla^b T_{ab} = 0$$

Energy density conservation:

$$\dot{\rho} + \theta(\rho + P) + D_a q^a = 0$$

Momentum conservation:

$$\dot{q}_a + \frac{4}{3}\theta q_a + (\rho + P)\dot{u}_a + D^b \pi_{ab} - D_a p = 0$$

For perfect fluid:

$$\dot{\rho} + \theta(\rho + P) = 0$$

$$(\rho + P)\dot{u}_a - D_a p = 0$$

Gravitational Waves

$$\nabla^d C_{abcd} = \nabla_{[b} R_{a]c} + \frac{1}{6} g_{c[b} \nabla_{a]} R$$

$\dot{E}_{ab}$ equation:

$$\dot{E}_{ab} + \theta E_{ab} + D^c B_{d(a} \eta_{b)ce}{}^{d} u^e + \frac{\kappa}{6\varphi}[3(\rho + P)\sigma_{ab} + 3D_{<a} q_{b>} - 3\dot{\pi}_{ab} - \theta \pi_{ab}]$$
$$+\frac{1}{2}\sigma_{ab}(\omega + \frac{3}{2})\frac{\dot{\varphi}^2}{\varphi^2} - \frac{1}{6}\frac{\sigma_{ab}}{\varphi} D_\mu D^\mu \varphi + \frac{1}{2}(\omega + \frac{3}{2})\frac{\dot{\varphi}}{\varphi^2} D_{<a} D_{b>} \varphi$$
$$+\frac{1}{2}\frac{\dot{\varphi}}{\varphi} E_{ab} + \frac{3}{4}\kappa \frac{\dot{\varphi}}{\varphi^2}\pi_{ab} = 0$$

$\dot{B}_{ab}$ equation:

$$\dot{B}_{ab} + (\theta + \frac{\dot{\varphi}}{2\varphi}) B_{ab} - (D^c E_{d(a} + \frac{\kappa}{2\varphi} D^c \pi_{d(a} + \frac{1}{2\varphi} D^c D_d D_{(a} \varphi$$
$$-\frac{1}{6\varphi} D^c D_\mu D^\mu \varphi \, h_{d(a}\,) \eta_{b)ce}{}^{d} u^e = 0$$

Gravitational Waves

$$\nabla^d C_{abcd} = \nabla_{[b} R_{a]c} + \frac{1}{6} g_{c[b} \nabla_{a]} R$$

div-E equation:

$$D^b E_{ab} - \frac{\kappa}{6\varphi}(2D_a\rho + 2\theta q_a + 3D^b\pi_{ab}) + \frac{2\kappa}{3}\rho\frac{D_a\varphi}{\varphi^2} - \frac{\kappa}{2}\frac{\dot{\varphi}}{\varphi^2}q_a + \frac{G}{3}\frac{D_a\varphi}{\varphi^2}V(\varphi)$$
$$-\frac{G}{6}\frac{D_a V(\varphi)}{\varphi} - (\frac{\omega}{3} + \frac{1}{2})\frac{\dot{\varphi}}{\varphi^2}[\frac{4}{3}\theta D_a\varphi + (D_a\varphi)^{\cdot} + \dot{u}\ \dot{\varphi}] = 0$$

div-B equation:

$$D^b B_{ab} - \frac{\kappa}{2\varphi}[(\rho + P)\eta_{ab}{}^{cd}u^b\omega_{cd} + \eta_{abcd}u^b D^c q^d] - \frac{1}{2}[(\omega\frac{\dot{\varphi}^2}{\varphi^2} - \frac{1}{3\varphi}D_\mu D^\mu\varphi$$
$$-\frac{\theta}{3}\frac{\dot{\varphi}}{\varphi} + \frac{\ddot{\varphi}}{\varphi})\eta_{ab}{}^{cd}u^b\omega_{cd} + \eta_{abcd}u^b(\omega\frac{\dot{\varphi}}{\varphi^2}D^c D^d\varphi + \frac{1}{\varphi}(D^c D^d\varphi)^{\cdot}$$
$$+\frac{\theta}{3\varphi}D^c D^d\varphi + \frac{\dot{\varphi}}{\varphi}D^c\dot{u}^d)] = 0$$

(Ⅲ)
CMB and LSS of Brans-Dicke Theory

Codes For CMB anisotropies of Brans-Dicke theory

CMBFAST	CAMB
Synchronous gauge	Covariant approach
by Xue-Lei Chen et al. PRD60,104036 (1999)	by Feng-Quan Wu et al. (2008)

Modified parts:

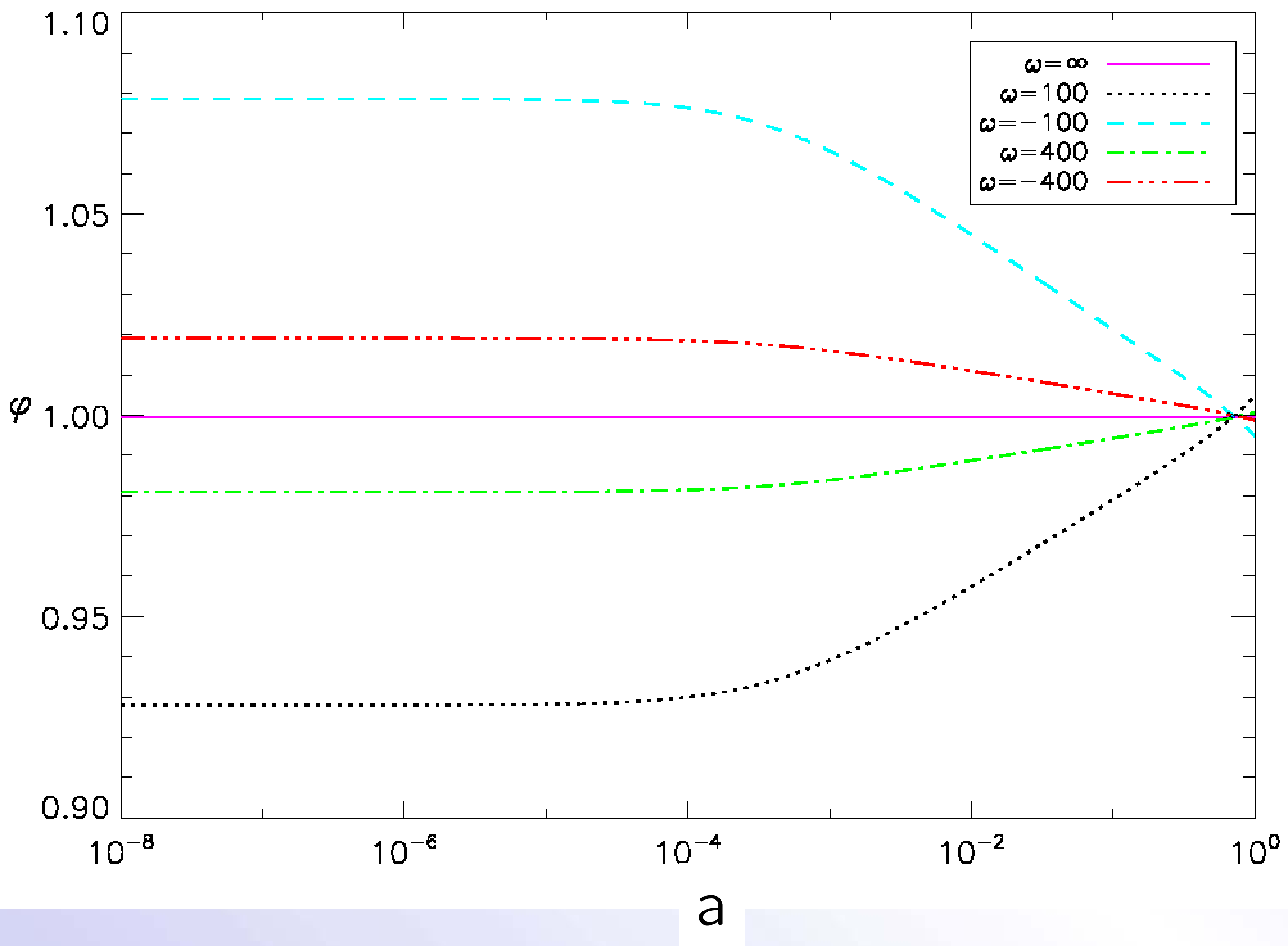
(1) Background evolution

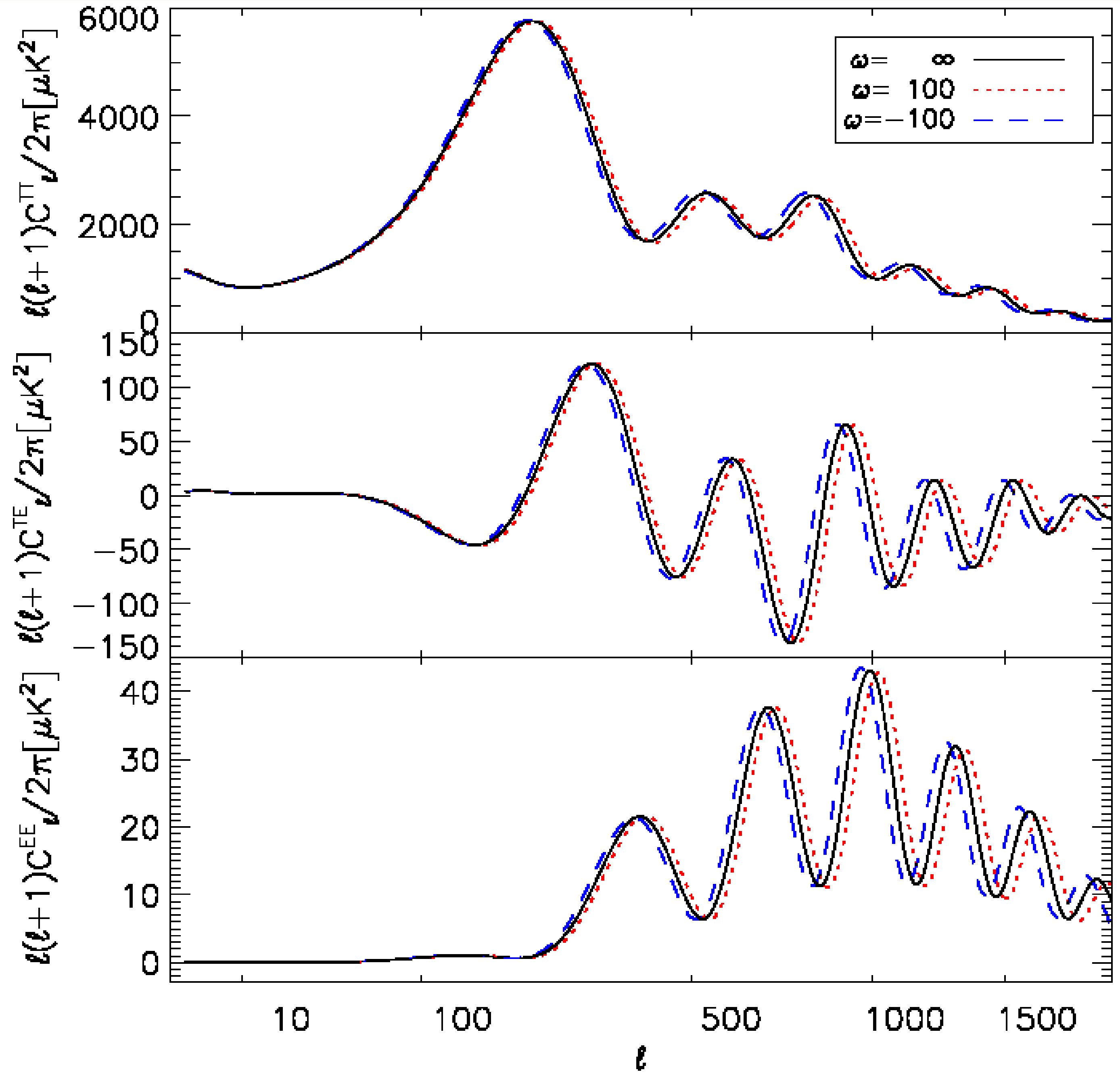
(2) Boltzmann-Einstein equations group

(3) Line-of-sight method

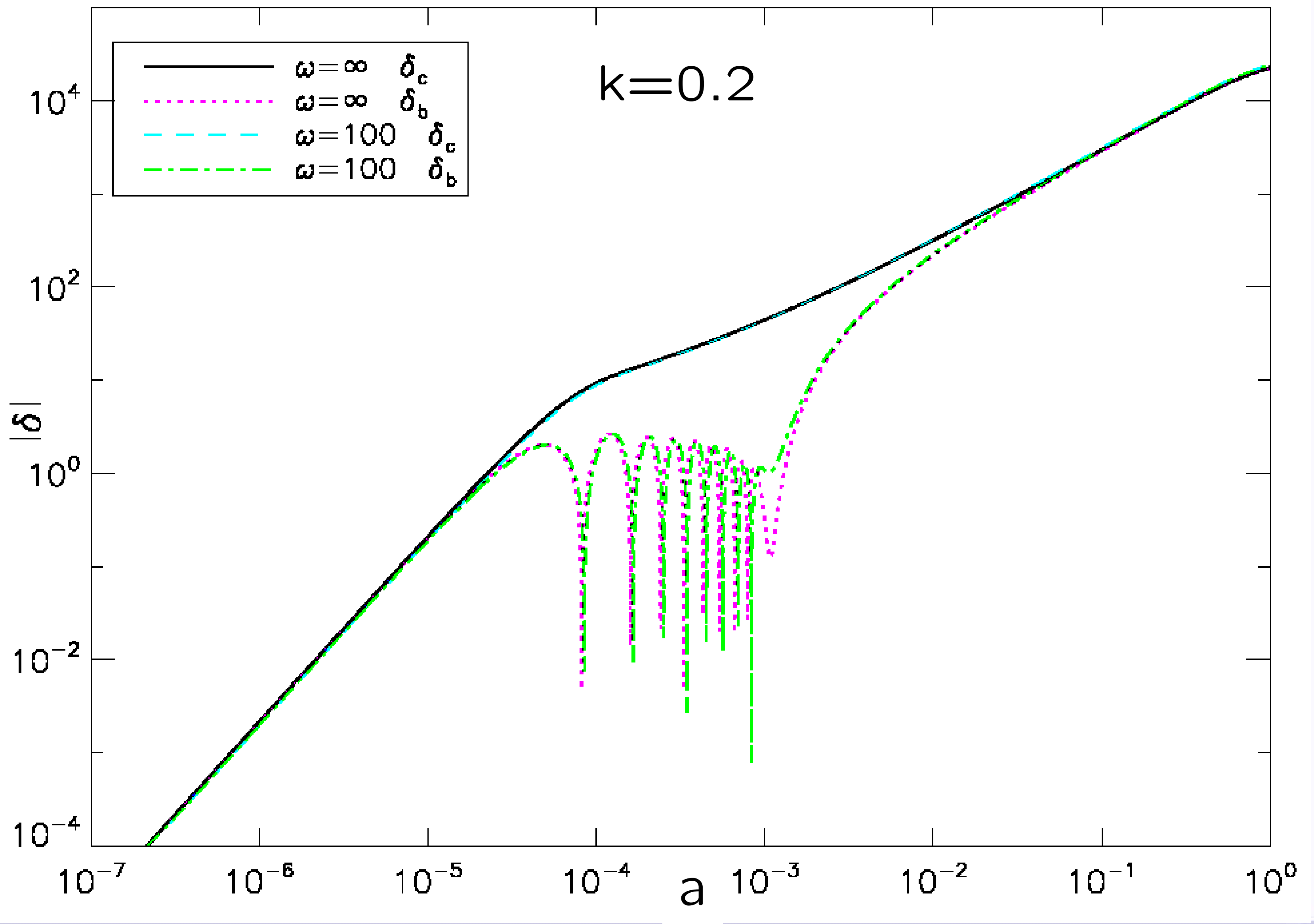
COSMOMC

Evolution of Brans-Dicke field

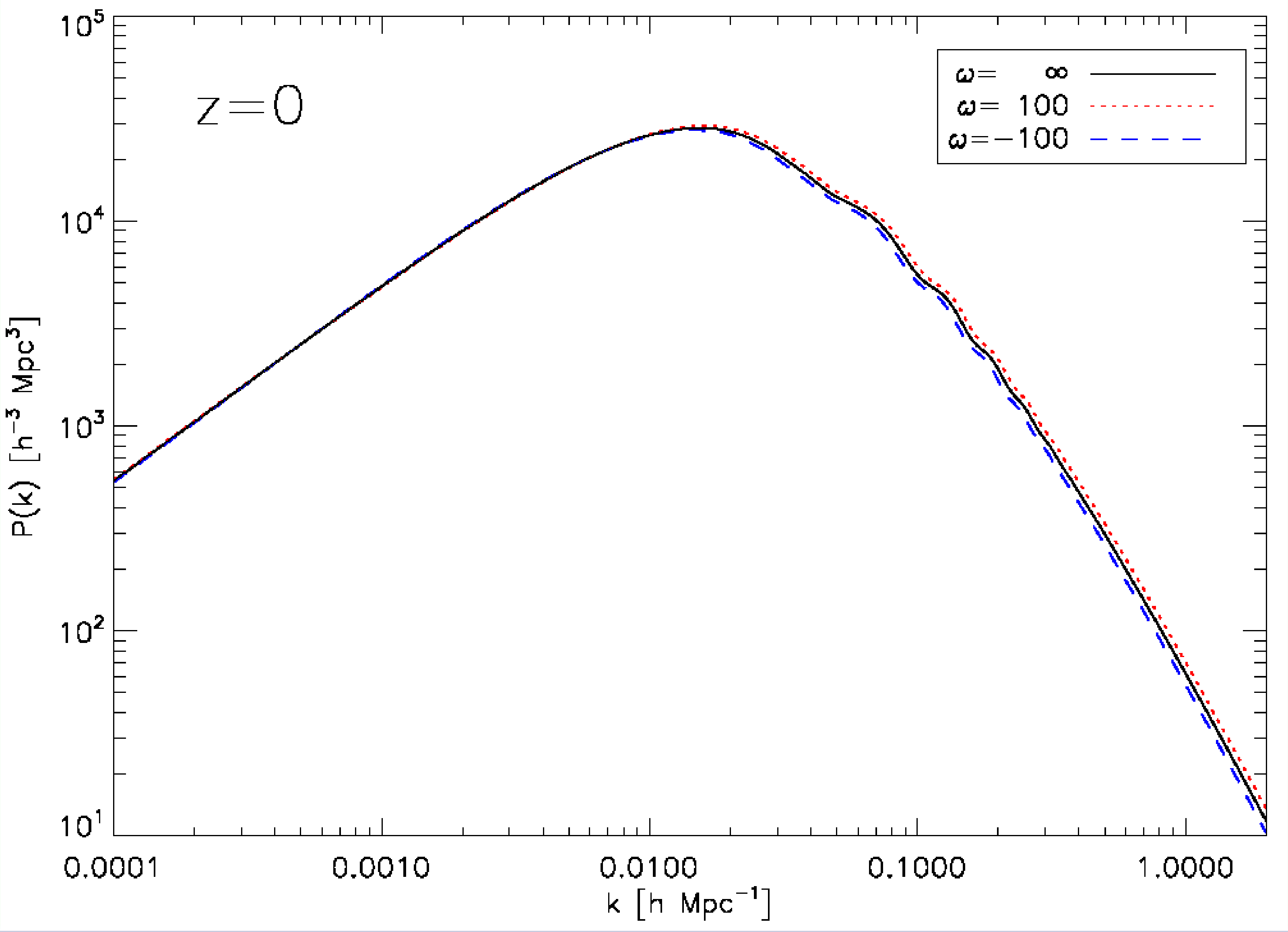



ω= ∞
ω= 100
ω=−100
ℓ(ℓ+1)C^TT_ℓ/2π[μK²]
ℓ(ℓ+1)C^TE_ℓ/2π[μK²]
ℓ(ℓ+1)C^EE_ℓ/2π[μK²]
6000
4000
2000
0
150
100
50
0
−50
−100
−150
40
30
20
10
0
10
100
500
1000
1500
ℓ

Evolution of density perturbations



Matter Power Spectra



(Ⅳ)
Cosmological Constraints on Brans-Dicke Theory

Constraint from Observations

(1) Solar system experiment: Cassini spacecraft

PPN parameters $\gamma = (1+\omega)/(2+\omega)$

$$\omega > 4 \times 10^4$$

$$G_{eff}(\varphi) = \frac{G}{\varphi}$$

(2) Cosmological observation:

2003 Nagata / 2004 Acquaviva — WMAP1 — CMBFAST — χ^2 / MCMC — 2σ — $\omega > 1000$ / $\omega > 120$

Fitting of parameters

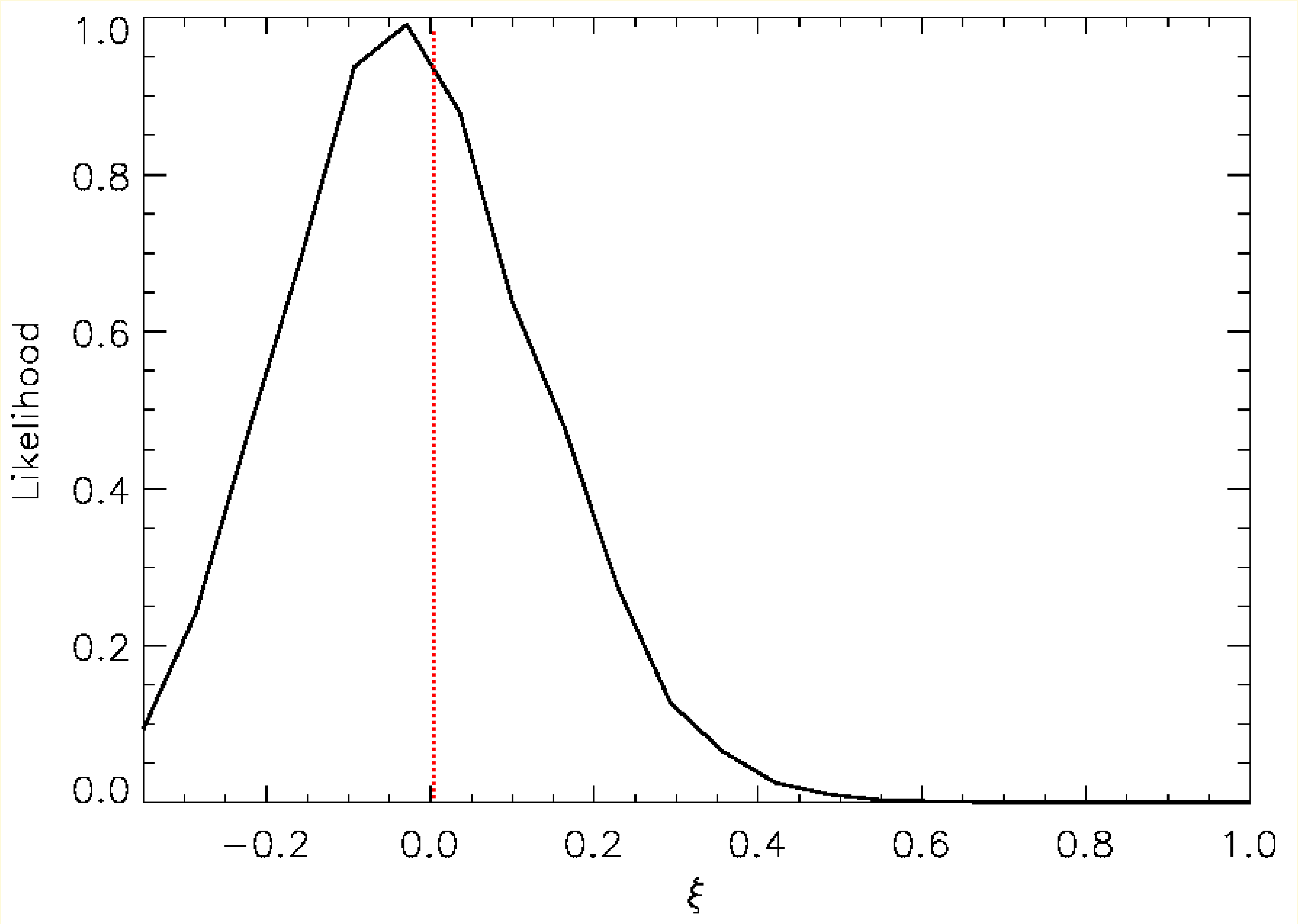
Data: { WMAP5 +acbar2007+CBI pol+B03
SDSS LRG DR4

Parameterization of ω for fitting:

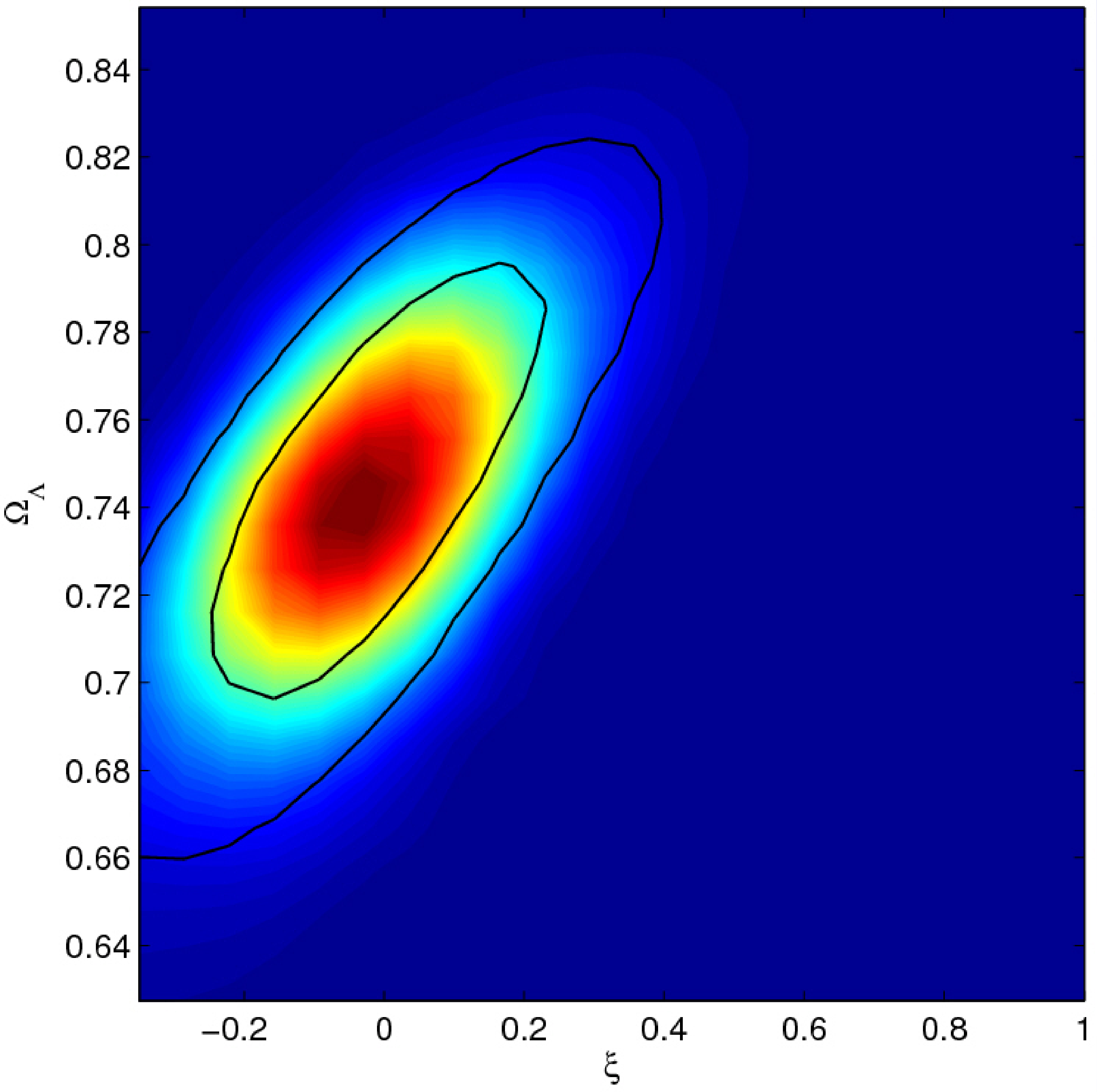
(1) $\xi = \frac{1}{\omega}$ 2σ confidence level: $\omega > 104$

(2) $\zeta = \ln \frac{1}{\omega}$ 2σ confidence level: $\omega > 116$

Marginalized Probability Distribution



Marginalized Probability Distribution



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Summary

Summary

(1) A full set of covariant and gauge-invariant perturbation theory for Brans-Dicke theory.

(2) Numerical codes for CMB anisotropy of Brans-Dicke theory

(3) Global fitting for parameters of Brans-Dicke theory.

(4) ⋯ scalar tensor theory ⋯ f(R) theory ⋯

Thank you!

Appendix

Brans-Dicke Theory

Action for the Brans-Dicke theory in the Jordan frame:

$$\mathcal{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[-\varphi R + \frac{\omega}{\varphi}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi\right] + \mathcal{S}^{(m)}$$

Generalized Einstein equations:

$$\begin{aligned} R_{ab} - \frac{1}{2}Rg_{ab} &= \frac{\kappa}{\varphi}T^{(m)}_{\mu\nu} + \frac{\omega}{\varphi^2}(\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}\nabla_\lambda\varphi\nabla^\lambda\varphi) \\ &\quad + g_{ab}\frac{GV(\varphi)}{2\varphi} + \frac{1}{\varphi}(\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\Delta\varphi) \end{aligned}$$

Equation of motion for φ :

$$\nabla_a\nabla^a\varphi = \frac{8\pi G}{2\omega+3}T^{(m)}$$

When $\omega \to \infty$, General Relativity is recovered.

Gravitational Waves

$$\nabla_{[e}R_{cd]ab} = 0$$

$$\nabla^d C_{abcd} = \nabla_{[b}R_{a]c} + \frac{1}{6}\, g_{c[b}\nabla_{a]}R$$

$$C_{abcd} = \left(g_{abqp}g_{cdsr} - \eta_{abqp}\eta_{cdsr}\right) u^q u^s E^{pr} - \left(\eta_{abqp}g_{cdsr} + g_{abqp}\eta_{cdsr}\right) u^q u^s H^{pr}$$

Matter Fields

Decomposition of the stress-energy tensor:

$$T_{ab} \equiv \rho u_a u_b + 2u_{(a} q_{b)} - P h_{ab} + \pi_{ab}$$

$$\rho \equiv T_{ab} u^a u^b$$

$$q_a \equiv h_a^b T_{bc} u^c$$

$$P \equiv -h^{ab} T_{ab}/3$$

$$\pi_{ab} \equiv T_{<ab>}$$

Spatial Curvature

3-Riemann tensor:

$$[D_a, D_b]u_c = {}^{(3)}R_{abdc}u^d$$

Spatial derivative of the projected Ricci scalar:

$$\eta_a \equiv \frac{S}{2} D_a {}^{(3)}R$$

For Brans-Dicke theory:

$$\begin{aligned}\eta_a \;=\;& \kappa\frac{\rho X_a}{\varphi} - \kappa\frac{\rho \mathcal{V}_a}{\varphi^2} - \frac{1}{S}(2\mathcal{H} + \frac{\varphi'}{\varphi})Z_a + \frac{1}{S^2}(\omega\frac{\varphi'}{\varphi^2} - \frac{3\mathcal{H}}{\varphi})(\mathcal{V}'_a - \mathcal{H}\mathcal{V}_a) \\ & +(\omega+3)\frac{1}{S^2}\frac{\varphi'}{\varphi^2}\mathcal{H}\mathcal{V}_a + \frac{1}{S}(\omega\frac{\varphi'^2}{\varphi^2} - 3\mathcal{H}\frac{\varphi'}{\varphi})W_a + S\frac{G}{2}\frac{dV(\varphi)}{d\varphi}\frac{D_a\varphi}{\varphi} \\ & -\frac{GV(\varphi)}{2\varphi^2}\mathcal{V}_a - \frac{\omega}{S^2}\frac{\varphi'^2}{\varphi^3}\mathcal{V}_a - \frac{1}{\varphi}D_a D_\nu \mathcal{V}^\nu - \frac{3}{S}\mathcal{H}^2\frac{\mathcal{V}_a}{\varphi}\end{aligned}$$

The Perturbation Variables

Density perturbation variable:

$$\delta = \frac{\delta\rho}{\rho}$$

Comoving fractional projected gradient of the density field:

$$X_a \equiv \frac{S}{\rho} D_a \rho \qquad \text{where} \qquad D_a = h_a^b \nabla_b$$

Comoving spatial gradient of the expansion rate:

$$Z_a \equiv S D_a \theta$$

Spatial derivative of the Brans-Dicke field:

$$\mathcal{V} \equiv S D_a \varphi$$

Harmonic Expansion

Generalized Helmholtz equation

$$S^2 D^a D_a Q^{(k)} = k^2 Q^{(k)}$$

$$Q^{(k)}_{A_l} = \frac{S}{k} D_{<a_l} Q^{(k)}_{A_{l-1}>}$$

$$X_a^{(i)} = \sum_k k X_k^{(i)} Q_a^{(k)} , \tag{1}$$

$$Z_a = \sum_k \frac{k^2}{S} Z_k Q_a^{(k)} , \tag{2}$$

$$q_a^{(i)} = \rho^{(i)} \sum_k q_k^{(i)} Q_a^{(k)} , \tag{3}$$

$$v_a^{(i)} = \sum_k v_k^{(i)} Q_a^{(k)} , \tag{4}$$

$$\pi_{ab}^{(i)} = \rho^{(i)} \sum_k \pi_k^{(i)} Q_{ab}^{(k)} , \tag{5}$$

$$E_{ab} = \sum_k \frac{k^2}{S^2} \Phi_k Q_{ab}^{(k)} , \tag{6}$$

$$\sigma_{ab} = \sum_k \frac{k}{S} \sigma_k Q_{ab}^{(k)} , \tag{7}$$

$$A_a = \sum_k \frac{k}{S} W_k Q_a^{(k)} , \tag{8}$$

$$\mathcal{V}_a = \sum_k k \mathcal{V}_k Q_a^{(k)} , \tag{9}$$

$$\eta_a = \sum_k \frac{k^3}{S^2} \eta_k Q_a^{(k)} \tag{10}$$

For scalar perturbation:

$$D^a X_k^{(i)} = O(2)$$

$$B_{ab} = O(2)$$

$$\omega_{ab} = O(2)$$

Codes For Brans-Dicke theory

(1) Backgroud evolution

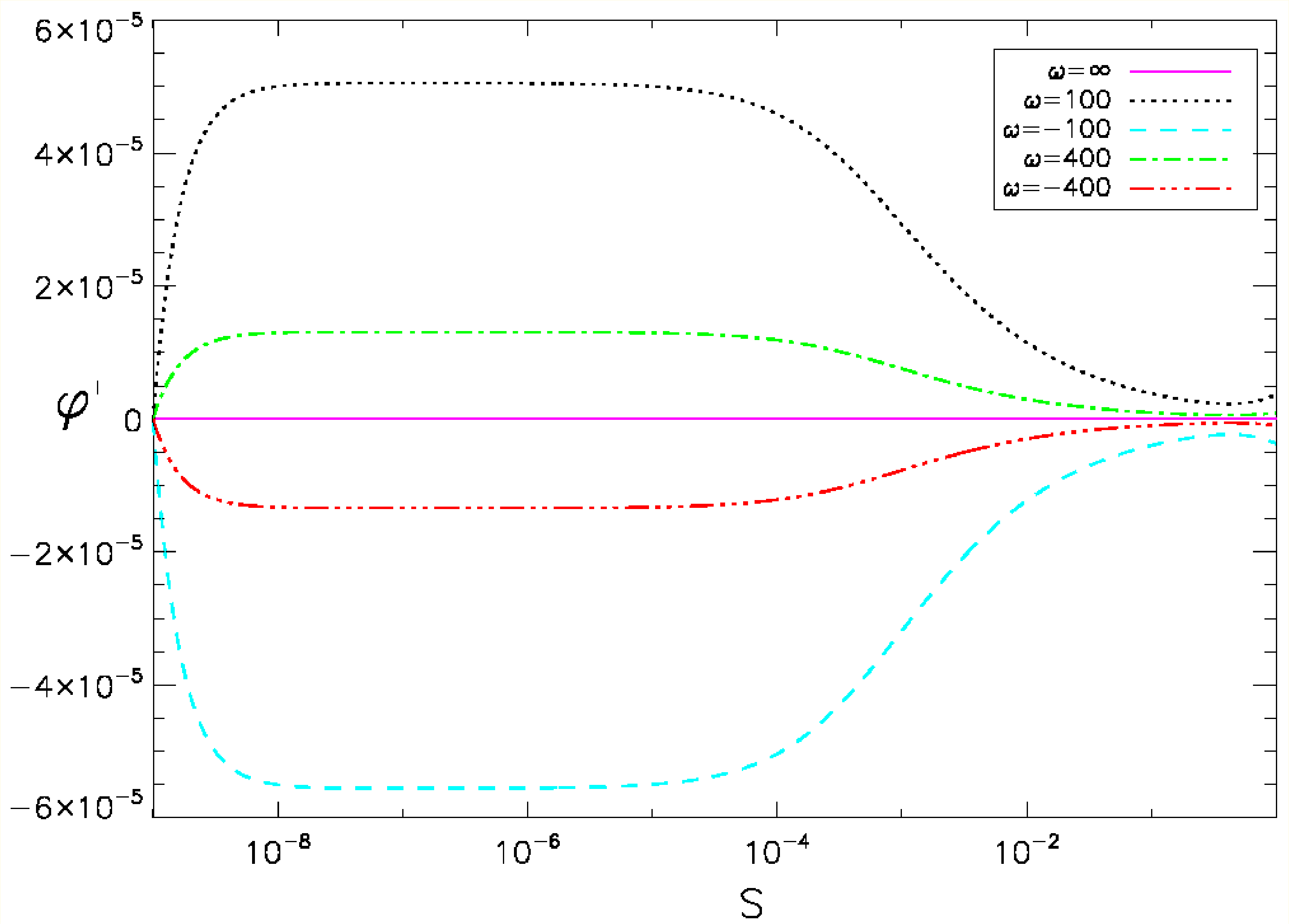
φ_i ⇐ $\varphi_0 = \dfrac{2\omega + 4}{2\omega + 3}$

$\varphi_i' = 0$ $\quad$ $\mathcal{V}_i = \mathcal{V}_i' = 0$ $\quad$ $\mathcal{V}_i = S D_a \varphi$

(2) Perturbation equation for $\mathcal{V}_i = S D_a \varphi$

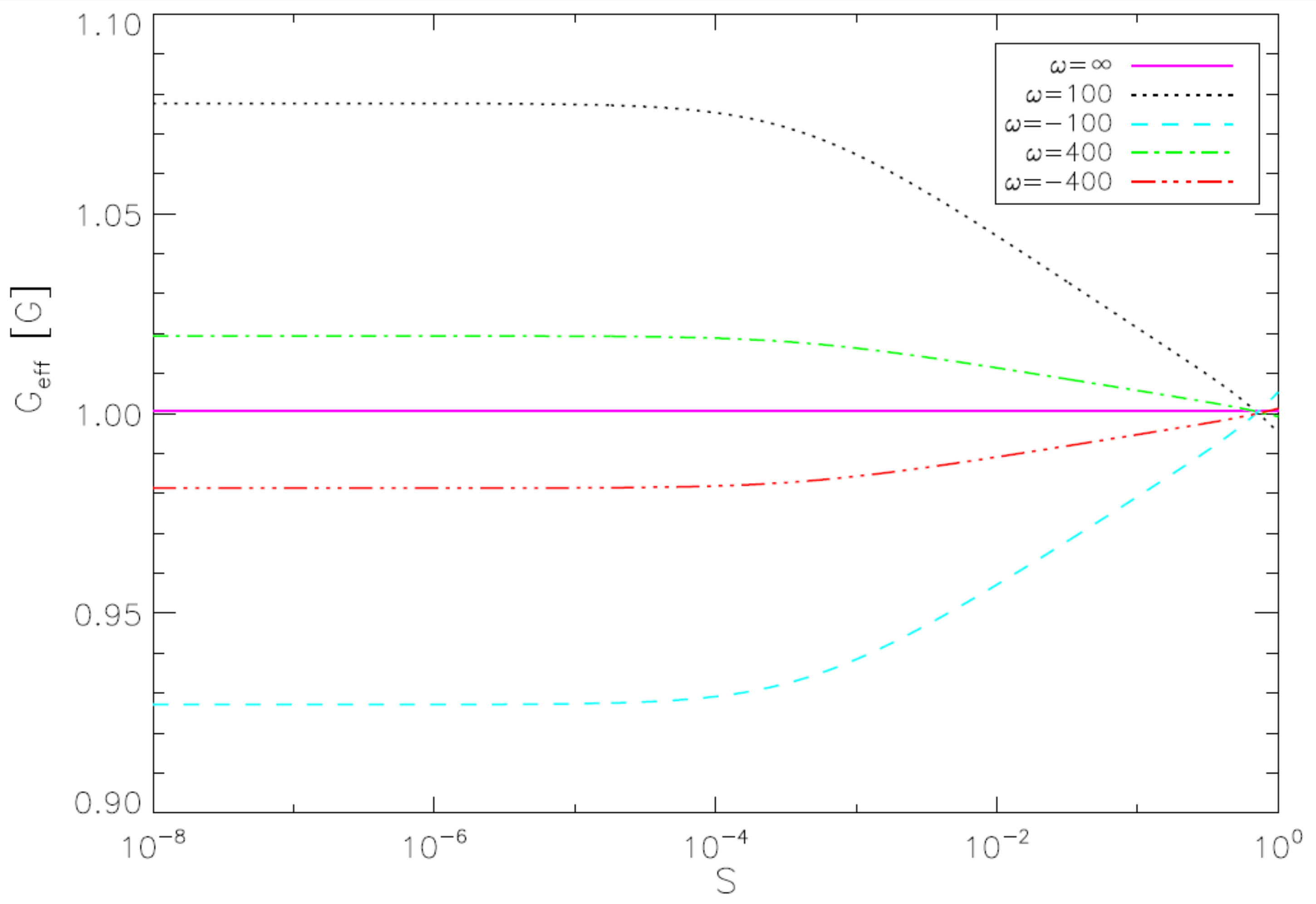
$$\mathcal{V}_a'' + 2\mathcal{H}\mathcal{V}_a' + S Z_a \varphi' + S^2 D_a D^b \mathcal{V}_b = \frac{\kappa S^2}{3 + 2\omega} \sum_i (1 - 3c_s^{(i)\,2}) \rho^{(i)} X^{(i)}$$

Evolution of Brans-Dicke field



Evolution of efficient Newtonian gravitational coupling

$$G_{eff}(\varphi) = \frac{1}{\phi} = \frac{G}{\varphi}$$



Fitting of parameters

Data: WMAP5 +acbar2007+CBI pol+B03
SDSS LRG DR4

Parameterization of ω for fitting:

(1) $\xi = \frac{25}{\omega}$ 2σ confidence level: $\omega > 104$

$\xi \in [-0.35, 1]$

$\omega \in [-\infty, -71] \cap [25, \infty]$

(2) $\xi = \ln \frac{25}{\omega}$ 2σ confidence level: $\omega > 116$

$\xi \in [-4.79, 0]$

$\omega \in [25, 3000]$